Topic 4 -

Inverse of a matrix

With numbers we have multiplicative inverses. For example, $3 \cdot \frac{1}{3} = 1$.

We write $3^{-1} = \frac{1}{3}$.

$\frac{1}{3}$ is the multiplicative inverse for 3.

What about for matrices?

<u>Def:</u> Let A be an $n \times n$ matrix. [So, A is a square matrix].

We say that A is <u>invertible</u> if there exists an $n \times n$ matrix B where $AB = BA = I_n$

If $AB = BA = I_n$, we say that A and B are <u>inverses</u> of each other.

Ex: Let

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}.$$

Let's check if A and B are inverses or not.

$$AB = \underbrace{\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}}_{2\times 2} \underbrace{\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}}_{2\times 2}$$

✓

answer $= 2\times 2$

$$= \begin{pmatrix} (1 \;\; 1)\cdot\begin{pmatrix} -1 \\ 2 \end{pmatrix} & (1 \;\; 1)\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ (2 \;\; 1)\cdot\begin{pmatrix} -1 \\ 2 \end{pmatrix} & (2 \;\; 1)\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -1+2 & 1-1 \\ -2+2 & 2-1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Also,

$$BA = \underbrace{\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}}_{2\times 2}\underbrace{\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}}_{2\times 2}$$

✓

answer is 2×2

$$= \begin{pmatrix} \begin{pmatrix} -1 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 2 \end{pmatrix} & \begin{pmatrix} -1 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 2 & -1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 2 \end{pmatrix} & \begin{pmatrix} 2 & -1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -1+2 & -1+1 \\ 2-2 & 2-1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I_2$$

Since $AB = BA = I_2$ we know A and B are inverses of each other.

Theorem: Suppose that A is an $n \times n$ matrix that is invertible, ie an inverse for A exists. Then there exists only one $n \times n$ matrix B that is the inverse of A, ie where

$$AB = BA = I_n.$$

Notation: If A is invertible, then we denote its unique inverse by A^{-1}.

Ex: $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$

We saw in the previous example that $A^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$

How to find A^{-1} for a square matrix A if it exists

Let A be an $n \times n$ matrix.

1. A^{-1} exists if and only if one can row reduce A down to I_n.

2. Procedure: Start with the matrix

$$(A \mid I_n)$$

Do row reduction on the above matrix until the left side is either I_n or has a row of zeros.

If you end up with a row zeros on the left side, A^{-1} does not exist.

If you end up with I_n on the left side then A^{-1} exists and its the matrix on the right side.

Ex: Find A^{-1}, if it exists, when $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$. ← 2x2

$$(A \mid I_2) = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right)$$

Goal: row reduce until left side is in reduced row echelon form

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right) \xrightarrow{-2R_1+R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{-R_2 \to R_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

row echelon form but not reduced row echelon form

$$\xrightarrow{-R_2+R_1 \to R_1} \left(\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right) = (R \mid B)$$

I_2 $\qquad$ A^{-1}

So, A^{-1} exists and $A^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$

Ex: Find A^{-1} if it exists

when
$$A = \begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}$$
3x3

$$\left(\begin{array}{ccc|ccc} \boxed{3} & 0 & 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array}\right)$$
$\underbrace{\qquad\qquad}_{A}$ $\underbrace{\qquad\qquad}_{I_3}$

Goal: Row reduce left side until either there is a row of zeros or I_3 is there

put a 1 here

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 3 & 0 & 3 & 1 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array}\right)$$

put zeros here

$$\xrightarrow[2R_1 + R_3 \to R_3]{-3R_1 + R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & \boxed{-3} & -3 & 1 & -3 & 0 \\ 0 & 5 & 4 & 0 & 2 & 1 \end{array}\right)$$

make this a 1

$$\xrightarrow{-\frac{1}{3}R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & \boxed{1} & 2 & 0 & 1 & 0 \\ 0 & \boxed{1} & 1 & -\frac{1}{3} & 1 & 0 \\ 0 & \boxed{5} & 4 & 0 & 2 & 1 \end{array}\right)$$

make these zeros

use this 1

$$\xrightarrow[-5R_2+R_3 \to R_3]{-R_2+R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/3 & 0 & 0 \\ 0 & 1 & 1 & -1/3 & 1 & 0 \\ 0 & 0 & \boxed{-1} & 5/3 & -3 & 1 \end{array}\right)$$

make this a 1

$$\xrightarrow{-R_3 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & \boxed{1} & 1/3 & 0 & 0 \\ 0 & 1 & \boxed{1} & -1/3 & 1 & 0 \\ 0 & 0 & \boxed{1} & -5/3 & 3 & -1 \end{array}\right)$$

use this 1 to

turn these into 0's

$$\xrightarrow[-R_3+R_2 \to R_2]{-R_3+R_1 \to R_1} \left(\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3} \;\middle|\; \underbrace{\begin{matrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{matrix}}_{A^{-1}}\right)$$

Since we were able to reduce the left side into I_3, the right side is A^{-1}.

So, $A^{-1} = \begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}$.

Ex: Find A^{-1} if it exists

when $A = \begin{pmatrix} 1 & 5 \\ -2 & -10 \end{pmatrix}$

make this a 0

$$\left(\underbrace{\begin{matrix} 1 & 5 \\ \boxed{-2} & -10 \end{matrix}}_{A} \;\middle|\; \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I_2}\right)$$

$$\xrightarrow{2R_1+R_2 \to R_2} \left(\begin{matrix} 1 & 5 \\ \underbrace{0 \quad 0}_{\text{row of zeros}} \end{matrix} \;\middle|\; \begin{matrix} 1 & 0 \\ 2 & 1 \end{matrix}\right)$$

So, A^{-1} does not exist

for $A = \begin{pmatrix} 1 & 5 \\ -2 & -10 \end{pmatrix}$.

HW 4 - Part 1

3 (b) Find A^{-1} if it exists

when $A = \begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}$

make this a 1

$$\left(\begin{array}{ccc|ccc} \boxed{-1} & 3 & -4 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right)$$

make these zeros

$$\xrightarrow{-R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ \boxed{2} & 4 & 1 & 0 & 1 & 0 \\ \boxed{-4} & 2 & -9 & 0 & 0 & 1 \end{array}\right)$$

make this a 1

$$\xrightarrow[4R_1 + R_3 \to R_3]{-2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & \boxed{10} & -7 & 2 & 1 & 0 \\ 0 & -10 & 7 & -4 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{1/10\, R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 1 & -7/10 & 1/5 & 1/10 & 0 \\ 0 & -10 & 7 & -4 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & \boxed{-3} & 4 & -1 & 0 & 0 \\ 0 & 1 & -7/10 & 1/5 & 1/10 & 0 \\ 0 & \boxed{-10} & 7 & -4 & 0 & 1 \end{array}\right)$$

make into zeros

$$\xrightarrow[10R_2 + R_3 \to R_3]{3R_2 + R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 19/10 & -2/5 & 3/10 & 0 \\ 0 & 1 & -7/10 & 1/5 & 1/10 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right)$$

row of zeros

Thus, A^{-1} does not exist

when $A = \begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}$.

<u>Theorem:</u> Let A and B be $n \times n$ matrices that are both invertible [That is, A^{-1} and B^{-1} both exist.]

1. Then, AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

2. Also, A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

<u>Note:</u> $(AB)^{-1} \neq A^{-1}B^{-1}$

you have to flip the order because sometimes

$$B^{-1}A^{-1} \neq A^{-1}B^{-1}$$

There's another way to represent a system of linear equations.

Given the system

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array}\right\} (*)$$

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

The system $(*)$ can be represented by the matrix equation

$$\underbrace{A\vec{x}}_{\text{matrix multiplication}} = \vec{b}$$

Ex: Consider the system

$$\boxed{\begin{aligned} x + 2y &= 3 \\ 4x + 5y &= 6 \end{aligned}}$$

Let's make the matrix equation that represents the system.

Let

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

Let's look at $A\vec{x} = \vec{b}$.

$$\underbrace{\underbrace{\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}}_{2\times 2}\underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{2\times 1}}_{\text{answer is } 2\times 1} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} \longleftarrow A\vec{x} = \vec{b}$$

This becomes

$$\underbrace{\begin{pmatrix} (1 \;\; 2) \cdot \begin{pmatrix} x \\ y \end{pmatrix} \\ \\ (4 \;\; 5) \cdot \begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix}}_{2 \times 1} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

This becomes

$$\boxed{\begin{pmatrix} x + 2y \\ 4x + 5y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}} \leftarrow A\vec{x} = \vec{b}$$

This is the same as

$$\boxed{\begin{aligned} x + 2y &= 3 \\ 4x + 5y &= 6 \end{aligned}}$$

Ex: Consider the system

$$\boxed{\begin{aligned} x + 4y - 2w + z &= 1 \\ 2x \qquad + w \qquad &= 3 \\ 14y - 12w + 7z &= 0 \end{aligned}}$$

Let

$$A = \begin{array}{c} \begin{pmatrix} 1 & 4 & -2 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 14 & -12 & 7 \end{pmatrix} \\ \begin{matrix} \underbrace{\quad}_{x} & \underbrace{\quad}_{y} & \underbrace{\quad}_{w} & \underbrace{\quad}_{z} \end{matrix} \end{array}, \quad \vec{x} = \begin{pmatrix} x \\ y \\ w \\ z \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

Let's look at $A\vec{x} = \vec{b}$:

$$\underbrace{\begin{pmatrix} 1 & 4 & -2 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 14 & -12 & 7 \end{pmatrix}}_{3\times 4} \underbrace{\begin{pmatrix} x \\ y \\ w \\ z \end{pmatrix}}_{4\times 1} = \underbrace{\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}}_{3\times 1}$$

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answer is 3×1

This becomes

$$\begin{pmatrix} (1\ \ 4\ \ -2\ \ 1)\cdot\begin{pmatrix} x\\ y\\ w\\ z\end{pmatrix} \\ (2\ \ 0\ \ 1\ \ 0)\cdot\begin{pmatrix} x\\ y\\ w\\ z\end{pmatrix} \\ (0\ \ 14\ \ -12\ \ 7)\cdot\begin{pmatrix} x\\ y\\ w\\ z\end{pmatrix}\end{pmatrix} = \begin{pmatrix} 1\\ 3\\ 0\end{pmatrix}$$

This becomes

$$\boxed{\begin{pmatrix} x+4y-2w+z \\ 2x+0y+w+0z \\ 0x+14y-12w+7z\end{pmatrix} = \begin{pmatrix} 1\\ 3\\ 0\end{pmatrix}}$$

This is the same as

$$\boxed{\begin{aligned} x+4y-2w+z&=1\\ 2x\qquad\quad +w\qquad&=3\\ 14y-12w+7z&=0\end{aligned}}$$

Theorem: Let A be an $n \times n$ matrix.
Suppose A^{-1} exists.
Then for each vector $\vec{b}$ in $\mathbb{R}^n$ there exists exactly one solution to the equation $A\vec{x} = \vec{b}$.
This solution is $\vec{x} = A^{-1}\vec{b}$.

proof: Suppose A^{-1} exists.

Then

$$A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = I_n \vec{b} = \vec{b}$$

Thus, $\vec{x} = A^{-1}\vec{b}$ solves $A\vec{x} = \vec{b}$.
Why is that the only solution?
Suppose you had a solution

$\vec{x}_0$ to $A\vec{x} = \vec{b}$.

So, $A\vec{x}_0 = \vec{b}$

Multiply both sides by A^{-1} on the left to get

$$A^{-1}(Ax_0) = A^{-1}\vec{b}$$

Thus,

$$\underbrace{(A^{-1}A)}_{I_n} x_0 = A^{-1}\vec{b}$$

So,

$$\underbrace{I_n x_0}_{x_0} = A^{-1}\vec{b}$$

Thus, $x_0 = A^{-1}\vec{b}$.

So the only solution is $A^{-1}\vec{b}$. $\blacksquare$

Ex: Find all the solutions to

$$\boxed{\begin{aligned} 3x \qquad\quad + 3z &= 9 \\ x + y + 2z &= -4 \\ -2x + 3y \qquad &= 5 \end{aligned}} \quad (*)$$

We can re-write this in matrix form.

Let

$$A = \begin{pmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}$$

The system $(*)$ becomes

$$\boxed{A\vec{x} = \vec{b}}$$

On Monday we showed that

$$A^{-1} = \begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}$$

Since A^{-1} exists, the system (*) will have exactly one solution and that solution will be

pg 1

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{x} = A^{-1}\vec{b}$$

$$= \underbrace{\begin{pmatrix} 2 & -3 & 1 \\ 4/3 & -2 & 1 \\ -5/3 & 3 & -1 \end{pmatrix}}_{3\times 3} \underbrace{\begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix}}_{3\times 1}$$

answer is 3×1

$$= \begin{pmatrix} \begin{pmatrix} 2 & -3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix} \\ \begin{pmatrix} 4/3 & -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix} \\ \begin{pmatrix} -5/3 & 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ -4 \\ 5 \end{pmatrix} \end{pmatrix} =$$

22

$$= \begin{pmatrix} 18+12+5 \\ 12+8+5 \\ -15-12-5 \end{pmatrix} = \begin{pmatrix} 35 \\ 25 \\ -32 \end{pmatrix}$$

Thus,

$$\boxed{x = 35, \; y = 25, \; z = -32}$$

is the only solution to

$$\boxed{\begin{aligned} 3x \quad\quad\quad + 3z &= 9 \\ x + y \quad + 2z &= -4 \\ -2x + 3y \quad\quad\quad &= 5 \end{aligned}}$$